Distortion of quasiconformal mappings with identity boundary values

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Workshop on Modern Trends
in Classical Analysis and Applications

The First Chinese-Finnish Seminar

August 17, 2012, Turku, Finland

Abstract

For a domain $D \subset \mathbb{R}^n$, $n \geq 2$, we consider the class of all K- quasiconformal maps of D onto itself with identity boundary values and Teichmüller's problem in this context. Given a map f of this class and a point $x \in D$, we show that the maximal dilatation of f has a lower bound in terms of the distance of x and $f(x)$ in the distance ratio metric. For instance, convex domains, bounded domains and domains with uniformly perfect boundaries are studied.

M. Vuorinen and X. Zhang, Distortion of quasiconformal mappings with identity boundary values. arXiv:1203.0427v1 [math.CV]

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Teichmüller's classical mapping problem

Finding a lower bound for the maximal dilatation of a QC self-homeo. which keeps the boundary pointwise fixed, and maps a given point of the domain to another given point of the domain.

Let D be a proper subdomain of $\mathbb{R}^n$ ($n \geq 2$), and let

$$\mathrm{Id}_K(\partial D) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ is } K\text{−quasiconformal} : f(x) = x, \forall x \in \mathbb{R}^n \setminus D\}.$$

O. Teichmüller, 1944

Let $D = \mathbb{R}^2 \setminus \{0, e_1\}$, $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$,

$$s_D(x, f(x)) \leq \log K$$

where s_D is the hyperbolic metric of $D = \mathbb{R}^2 \setminus \{0, e_1\}$.

The above result may be considered as a stability result since $f(x)$ is contained in the closure of the hyperbolic ball $B_{s_D}(x, \log K)$ with the radius tending to 0 as $K \to 1$.

Reshetnyak's stability theory [R]:

- Liouville's theorem
- estimate of the distance of a K-qc map from the "nearest" Möbius transformation in a suitable norm. normal family, lack of explicit estimates.

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Asymptotically sharp explicit bounds for the convergence of K-qc maps to the case $K = 1$. There are very few of these in the literature. The key results are

Sharp dimension-free Schwarz lemma [AVV,1986]

Let $f : \mathbb{B}^n \to f\mathbb{B}^n$ be $K-$quasiconformal with $f(0) = 0$ and $f\mathbb{B}^n \subset \mathbb{B}^n$. Then

$$|f(x)| \leq \lambda_n^{1-\alpha}|x|^{\alpha} \leq 2^{1-1/K}K|x|^{1/K}$$

for $x \in \mathbb{B}^n$, where $\alpha = K^{1/(1-n)}$ and λ_n is constant depending only on n. For each $n \geq 2$ the inequality is sharp for $K = 1$.

Let $\eta_{K,n}(t) = \sup\{|f(x)| : |x| = t,\ f \in QC_K(\overline{\mathbb{R}^n}),\ f(e_1) = e_1\}$ for $t > 0$, where $e_1 = (1, 0, \ldots, 0)$.

Explicit estimate of the function of quasisymmetry of K-QC maps [Vu1, 1990]

The following inequalities hold for $n \geq 2$ and $K > 1$:

- $\eta_{K,n}(1) \leq \exp\{6(K+1)^2\sqrt{K-1}\}$;
- $\eta_{K,n}(t) \leq \eta_{K,n}(1)\varphi_{K,n}(t), \quad 0 \leq t \leq 1,$
  $\eta_{K,n}(t) \leq \eta_{K,n}(1)\varphi_{1/K,n}(1/t), \quad t \geq 1.$

Typically proofs make use of conformal invariants and moduli of curve families and sometimes involve special functions. Ideologically, our results follow the approach based on explicit asymptotic sharp estimates.

The hyperbolic metric $\rho_{\mathbb{B}^n}(x,y)$ on $\mathbb{B}^n$:

$$\tanh^2 \frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)}.$$

The *quasihyperbolic metric* k_D:

$$k_D(x,y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{1}{d(z)} |dz|, \quad x, y \in D,$$

where Γ is the family of all rectifiable curves in D joining x and y, and $d(z) = d(z, \partial D)$ is the Euclidean distance between z and the boundary of D.

The *distance-ratio metric* or *j−metric*:

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right), \quad x, y \in D.$$

$$j_D(x,y) \le k_D(x,y)$$

uniform domain D: $\exists U = U(D) \ge 1$ s.t. $k_D(x,y) \le U\, j_D(x,y)$ for all $x, y \in D$.

The *Grötzsch ring domain* $R_{G,n}(s)$, $s > 1$, and the *Teichmüller ring domain* $R_{T,n}(t)$, $t > 0$, are doubly connected domains with complementary components $(\overline{\mathbb{B}^n}, [se_1, \infty)$ and $([-e_1, 0], [te_1, \infty))$, respectively. For their capacities we write

$$\begin{cases} \gamma_n(s) = \operatorname{cap} R_{G,n}(s) = M(\Delta(\overline{\mathbb{B}^n}, [se_1, \infty])), \\ \tau_n(t) = \operatorname{cap} R_{T,n}(t) = M(\Delta([-e_1, 0], [te_1, \infty])). \end{cases}$$

Functional identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1).$$

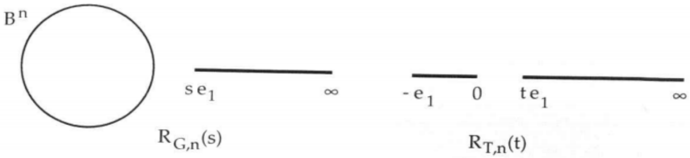

Figure: Grötzsch and Teichmüller rings

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}, \quad 0 < r < 1.$$

$\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$

$$r^\alpha \le \varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^\alpha \le 2^{1-1/K} K r^\alpha, \quad \alpha = K^{1/(1-n)},$$

$$2^{1-K} K^{-K} r^\beta \le \lambda_n^{1-\beta} r^\beta \le \varphi_{1/K,n}(r) \le r^\beta, \quad \beta = 1/\alpha,$$

where $K \geq 1, r \in (0,1)$, and the constant $\lambda_n \in [4, 2e^{n-1})$ is the so-called *Grötzsch ring constant*.
For $n \geq 2$, $t \in (0,\infty)$, $K > 0$, we denote

$$\eta_{K,n}(t) = \tau_n^{-1}\left(\frac{1}{K}\tau_n(t)\right) = \frac{1-\varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2}.$$

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Let $\alpha > 0$ and assume that $D \subset \overline{\mathbb{R}^n}$ is a closed set containing at least two points. Then D is $s-$*uniformly perfect* if there is no ring domain separating D with the modulus greater than s.

Aseev, Sibrian Math J, 1999

Suppose that $s > 0$ and that $s-$uniformly perfect sets E_0 and E_1 meets each component of the complement of the spherical ring $D = \{x : r_1 < |x - x_0| < r_2\} \subset \mathbb{R}^n$ with the following relation between the radii

$$r_2/r_1 > 1 + 2e^s.$$

Then

$$\mathrm{cap}(E_0, E_1; D) \geq C \log \frac{r_2}{r_1},$$

where the constant $C > 0$ depends only on s and the dimension n of the space.

- J. Krzyż, 1968: $D = \mathbb{B}^2$
- G.D. Anderson, M.K. Vamanamurthy, 1979:
  $D = \mathbb{B}^n, \quad n \geq 3$, (additional symmetry hypothesis)
- V. Manojlović, M. Vuorinen, 2011: $D = \mathbb{B}^n, \quad n \geq 3$

[MV], $D = \mathbb{B}^n, \quad n \geq 3$

If $f \in \mathrm{Id}_K(\partial \mathbb{B}^n)$, then for all $x \in \mathbb{B}^n$,

$$\rho_{\mathbb{B}^n}(x, f(x)) \leq \log\left(1 + \frac{1-2a}{a}\right), \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2,$$

where $\rho_{\mathbb{B}^n}$ is the hyperbolic metric of the unit ball, and $\varphi_{K,n}$ is as in .

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Vuorinen, 1984 [Vu2]: D =uniform domains with connected boundary

$$K \geq c_1(n, D) k_D(x, f(x))^n$$

whenever $k_D(x, f(x))$ exceeds a bound depending only on n and D. Here $c_1(n, D)$ is a positive constant depending only on n and D.

As pointed out in [Vu2], it is not true for $n \geq 3$ that $k_D(x, f(x)) > 0$ implies $K > 1$. Indeed, let $X = \{(x, 0, 0) : x \in \mathbb{R}\}$ be the x_1-axis, let $D = \mathbb{R}^3 \setminus X$, and let $f : D \to D$ be a rotation around the x_1-axis with $f(x) = (0, -1, 0), x = (0, 1, 0)$. Then f is conformal, i.e. $K = 1$, f keeps the x_1-axis $X = \partial D$ pointwise fixed, and D is a uniform domain with connected boundary X and $k_D(x, f(x)) = \pi$. Clearly, for this domain $c_1(3, D) \leq 1/\pi^3$.

For convex domains we can get an improved distortion theorem which shows that for each $x \in D$, the requirement $f(x) \neq x$ implies the maximal dilatation of f to be greater than 1. This kind of behavior also holds for bounded domains.

Convex domains

Let $D \subsetneq \mathbb{R}^n$ be a convex domain and $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$

$$j_D(x, f(x)) \leq \log\left(1 + \sqrt{c_2(n,K)^2 - 1}\right) \leq \log\left(1 + \frac{\sqrt{1-2a}}{a}\right)$$

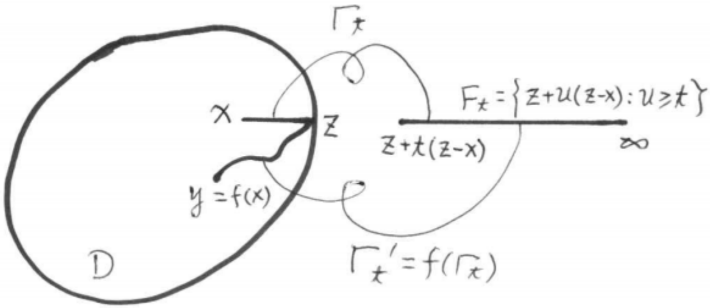
where $c_2(n,K) = \min\{\eta_{K,n}(1), \eta_{1/K,n}(1)^{-1}\}$ with $c_2(n,K) \to 1$ as $K \to 1$, and $a = \varphi_{1/K,n}(1/\sqrt{2})^2 \to 1/2$ as $K \to 1$. Here j_D is the distance ratio metric in D.

Proof. Write $y = f(x)$. We may assume $d(x) \le d(y)$ since $f^{-1} \in \mathrm{Id}_K(\partial D)$ also. Fix $z \in \partial D$ such that $d(x) = |x - z|$. For $t > 0$, write $F_t = \{z + u(z - x) : u \ge t\}$. Let $\Gamma_t = \Delta([x, z], F_t)$ be the family of all curves in $\mathbb{R}^n$ joining $[x, z]$ to F_t. $\Gamma'_t = f(\Gamma_t) = \Delta(f([x, z]), F_t)$. It follows that

$$\tau_n\left(\frac{t|x - z|}{|y - z|}\right) \le M(\Gamma'_t) \le K\, M(\Gamma_t) = K\,\tau_n(t).$$

Setting $t = 1$, we have

$$\frac{|y - z|}{|x - z|} \le \frac{1}{\tau^{-1}(K\tau_n(1))} = \frac{1}{\eta_{1/K,n}(1)},$$


Γt
Ft = {z+u(z-x): u≥t}
x
z
z+t(z-x)
∞
y = f(x)
D
Γt′ = f(Γt)

and setting $t = |y-z|/|x-z|$,

$$\frac{|y-z|}{|x-z|} \le \tau^{-1}\left(\frac{\tau_n(1)}{K}\right) = \eta_{K,n}(1).$$

Hence it follows that

$$\frac{|y-z|}{|x-z|} \le \min\{\eta_{K,n}(1), \frac{1}{\eta_{1/K,n}(1)}\} = c_2(n,K).$$

Since D is convex, it is easy to see that $|y-z|^2 \ge |x-y|^2 + |x-z|^2$, and hence

$$\frac{|x-y|}{|x-z|} \le \sqrt{\left(\frac{|y-z|}{|x-z|}\right)^2 - 1}\,.$$

The definition of the $j-$metric, together with the last two inequalities yields

$$j_D(x,y) \le \log\left(1 + \sqrt{c_2(n,K)^2 - 1}\right),$$

as desired.

For K close to 1, the above inequality can be simplified further.

Convex domain, K close to 1

Let $D \subsetneq \mathbb{R}^n$ be a convex domain and

$$K_n = \left(1 + \frac{\log 3}{2(n-1) + \log 8}\right)^{n-1} \in [K_2, \sqrt{3}), \qquad K_2 \approx 1.2693,$$

and let $K \in (1, K_n]$ and $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$

$$j_D(x, f(x)) \leq 4\sqrt{K-1}.$$

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By using Grötzsch's extremal ring we can get a slightly improved bounds for the case of unit ball and convex domains.

The unit ball

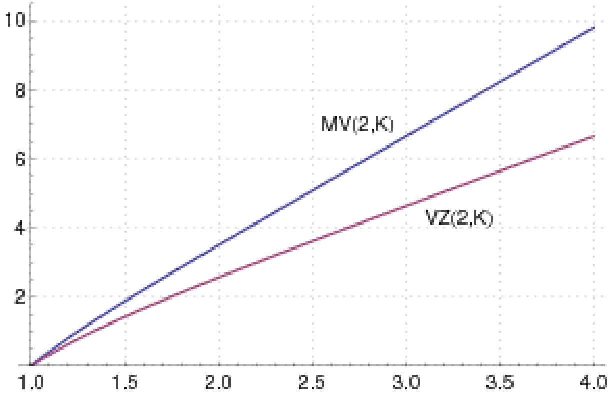
If $f \in \mathrm{Id}_K(\partial\mathbb{B}^n)$, then for all $x \in \mathbb{B}^n$,

$$\rho_{\mathbb{B}^n}(x, f(x)) \leq \log \frac{2\varphi_{K,n}(1/3)}{1-\varphi_{K,n}(1/3)}.$$

Convex domains

Let $D \subsetneq \mathbb{R}^n$ be a convex domain and $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$

$$j_D(x, f(x)) \leq \log\left(1+\sqrt{\left(\frac{2\varphi_{K,n}(1/3)}{1-\varphi_{K,n}(1/3)}\right)^2 - 1}\right).$$



10
8
6
4
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1.0
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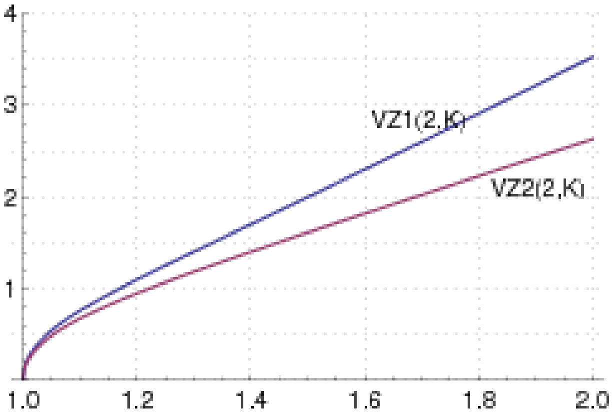
MV(2,K)
VZ(2,K)


4
3
2
1
1.0
1.2
1.4
1.6
1.8
2.0
VZ1(2,K)
VZ2(2,K)

Bounded domain

Let D be a bounded domain in $\mathbb{R}^n$, and $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$

$$|f(x) - x| \le \mathrm{diam}(D) \tanh\left(\frac{1}{2} \log \frac{1-a}{a}\right), \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2 .$$

Proof. For $x \in D$, $D \subset \mathbb{B}^n(x, \mathrm{diam}(D))$ since D is bounded. Let $g(w) = (w - x)/\mathrm{diam}(D)$, then $h = g \circ f \circ g^{-1} \in \mathrm{Id}_K(\partial \mathbb{B}^n)$. By MV's theorem,

$$\rho_{\mathbb{B}^n}\left(\frac{f(x) - x}{\mathrm{diam}(D)}, 0\right) = \rho_{\mathbb{B}^n}(h(0), 0) \le \log \frac{1-a}{a},$$

$$|f(x) - x| \le \mathrm{diam}(D) \tanh\left(\frac{1}{2} \log \frac{1-a}{a}\right)$$

since $\rho_{\mathbb{B}^n}(z, 0) = 2\mathrm{arctanh}|z|$ for $z \in \mathbb{B}^n$.

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The following theorem concerns the Hölder continuity of quasiconformal self mappings in $\mathrm{Id}_K(\partial\mathbb{B}^n)$

Hölder continuity

If $f \in \mathrm{Id}_K(\partial\mathbb{B}^n)$, then for all $x, y \in \mathbb{B}^n$

$$|f(x) - f(y)| \leq M_1(n,K)|x-y|^\alpha, \qquad \alpha = K^{1/(1-n)}$$

where $M_1(n,K) = \lambda_n^{1-\alpha} C(\alpha)$ and
$C(\alpha) = 2^{1-\alpha}\alpha^{-\alpha/2}(1-\alpha)^{(\alpha-1)/2}$, with $M_1(n,K) \to 1$ when
$K \to 1$, and $\lambda_n \in [4, 2e^{n-1})$ is the Grötzsch ring constant.

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Proof. For $R > 1$ let $h(x) = x/R$, then $g := h \circ f \circ h^{-1} : \mathbb{B}^n \to \mathbb{B}^n$ is a K-quasiconformal mapping. By applying the following well-known inequality

$$\tanh \frac{\rho(f(x), f(y))}{2} \leq \varphi_{K,n}\left(\tanh \frac{\rho(x,y)}{2}\right)$$

and the estimate for the hyperbolic metric

$$\frac{|x-y|}{1+|x||y|} \leq \tanh \frac{\rho(x,y)}{2} \leq \frac{|x-y|}{1-|x||y|}$$

to the mapping g and points $x/R, y/R$ for $x, y \in \mathbb{B}^n$, we have

$$\frac{|f(x)/R - f(y)/R|}{1+|f(x)||f(y)|/R^2} \leq \varphi_{K,n}\left(\frac{|x/R - y/R|}{1-|x||y|/R^2}\right).$$

Hence

$$\begin{aligned} |f(x)-f(y)| &\leq \lambda_n^{1-\alpha} \frac{R+|f(x)||f(y)|/R}{(R-|x||y|/R)^\alpha} |x-y|^\alpha \\ &\leq \lambda_n^{1-\alpha} A(R) |x-y|^\alpha. \end{aligned} \quad (1)$$

where

$$A(R) = \frac{R + R^{-1}}{(R - R^{-1})^{\alpha}}.$$

It is easy to check that $A(1+) = \infty = A(\infty)$ and

$$R_0 = \sqrt{\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}}$$

is the unique value of R in the interval $(1, \infty)$ such that $A'(R) = 0$. Hence we have

$$C(\alpha) := \min_{1<R<\infty} A(R) = A(R_0) = 2^{1-\alpha}\alpha^{-\alpha/2}(1-\alpha)^{(\alpha-1)/2}.$$

Since the inequality (1) holds for all $R > 1$, we get

$$|f(x) - f(y)| \le \lambda_n^{1-\alpha} C(\alpha)|x - y|^{\alpha}.$$

It is easy to see that $C(1-) = 1$, and hence $M_1(n, K) = \lambda_n^{1-\alpha} C(\alpha) \to 1$ as $K \to 1$.

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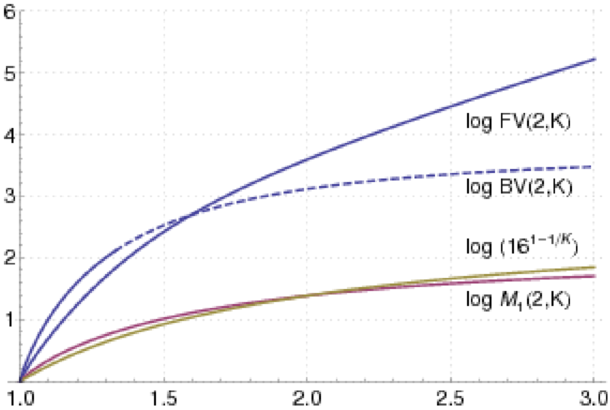
6
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2.0
2.5
3.0
log FV(2,K)
log BV(2,K)
log $(16^{1-1/K})$
log $M_1(2,K)$

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KIITOS!